PHYSICAL REVIEW E VOLUME 55, NUMBER 6 JUNE 1997

# Gibbsian versus non-Gibbsian nature of stationary states for Toom probabilistic cellular automata via simulations

Danuta Makowiec*
*Institute of Theoretical Physics and Astrophysics, Gdańsk University, ulica Wita Stwosza 57, 80-952 Gdańsk, Poland*

(Received 30 January 1997)

Ising-type behavior found in cellular automata with the majority vote rule hints at the possible relation between stationary states of these cellular automata and Gibbs states of some equilibrium statistical model. Results of computer experiments aimed at testing properties of stationary states for Toom probabilistic cellular automata such as decay of correlations, boundary configuration dependence, and large-fluctuation analysis are presented. By the cluster volume analysis the estimation for energy density carried by stationary states is proposed. [S1063-651X(97)00406-6]

PACS number(s): 05.50.+q, 05.70.Jk

## I. INTRODUCTION

The standard Toom probabilistic cellular automata are defined on the square lattice of two-level spins with local interactions being the perturbed majority vote of three spins: north, east, and center [1]. That is, the probability that a center spin $\sigma_{x,y}$ occupying an $(x,y)$ node on a square lattice takes at the next time the value $\sigma'_{x,y}$ depends on the spin states of its north neighbor $\sigma_{x-1,y}$, and its east neighbor $\sigma_{x,y+1}$, and itself as

$$\mathrm{Prob}\{(\sigma_{x,y},\sigma_{x-1,y},\sigma_{x,y+1})\rightarrow\sigma'_{x,y}\}$$
$$=\tfrac{1}{2}[1+(1-2\varepsilon)\sigma_{x,y}\mathrm{sgn}(\sigma_{x,y}+\sigma_{x-1,y}+\sigma_{x,y+1})]. \quad (1)$$

Thus, with probability $1-\varepsilon$, the dynamics follows the deterministic majority vote rule over the triangle formed by center, north, and east neighboring spins and with probability $\varepsilon$ opposes this rule. One can notice that the deterministic Toom rule, i.e., $\varepsilon=0$, becomes the Domany cellular automata rule for the zero temperature [2,3].

The Toom cellular automata are known from their critical behavior. The property that is characteristic of the first-order phase transition, namely, the discontinuity of the order parameter — magnetization — can be identified when one changes random initial configurations. For purely deterministic dynamics, if an initial configuration is random, i.e., spin states are chosen independently of each other with probability $p$ to obtain an up state, then one observes ergodic behavior for all $p$ enough far away from 1/2. The resulting stationary configuration is homogeneous with all spins *up* (for $p>1/2$) or all spins *down* (for $p<1/2$). If an initial configuration is prepared with $p$ being close to 1/2, then some other stationary configurations appear, partially due to the periodic boundary conditions of simulated systems. These stationary configurations are called *flat-interface* [4] or *mixed* [6] configurations because they consist of two homogeneous disjoint areas, the boundary of which is a straight line (vertical, horizontal, or diagonal along the NW-SE direction). It has been found in simulations that the system is chaotic there. A change of a single spin state can cause the cellular automata system to be attracted to different stationary configurations (that so-called damage-spreading study [5]; see [6] for details).

The probabilistic Toom model provides the possibility to study thermodynamics phenomena. In particular, stationary states of this model can be considered as candidates for equilibrium states of some thermodynamic system with the $\varepsilon$ parameter mimicking temperature effects. This way one can look for links between the stationarity of probabilistic cellular automata and some equilibrium systems [7,8].

In our experiments we always start with a system with all spins up (the magnetization is $+1$). The standard Monte Carlo procedure leads such a system to the stationary configurations corresponding to a given thermal perturbation. The stationary cellular automata states are represented as averages over thousands of stationary configurations obtained in many steps of discrete evolution. The size of the lattices considered in experiments varies from $L=15$ (in the finite-lattice-size scaling) to $L=500$ (the locality study), depending on the particular experimental demands, and thermalization time is equal to $50L$.

With our results we will discuss the following problem: whether or not the stationary configurations arising on a spin lattice from the probabilistic cellular automata evolution, i.e., states represented by an invariant measure with respect to the probabilistic cellular automata stochastic transformation, possess the Gibbsian nature. By the Gibbsian nature of any probabilistic measure $\mu(\sigma)$ we mean that we can associate some "reasonable" function $\mathcal{H}(\sigma)$ [4] that represents energy carried by a configuration $\sigma$. Formally, the Hamiltonian $\mathcal{H}$ arising from the Gibbs measure $\mu$ reads

$$\mathcal{H}(\sigma)=-\ln\mu(\sigma)+\mathrm{const}, \quad (2)$$

where the temperature parameter is included in the Hamiltonian. To give sense to the formal formula (2) one needs to know how the probability of any finite volume configuration $\sigma_\Lambda$ depends on the boundary configuration $\sigma_{\Lambda^c}$, i.e., $\mathrm{Prob}\{(\sigma_\Lambda|\sigma_{\Lambda^c})\}$.

*Electronic address: fizdm@univ.gda.pl

 

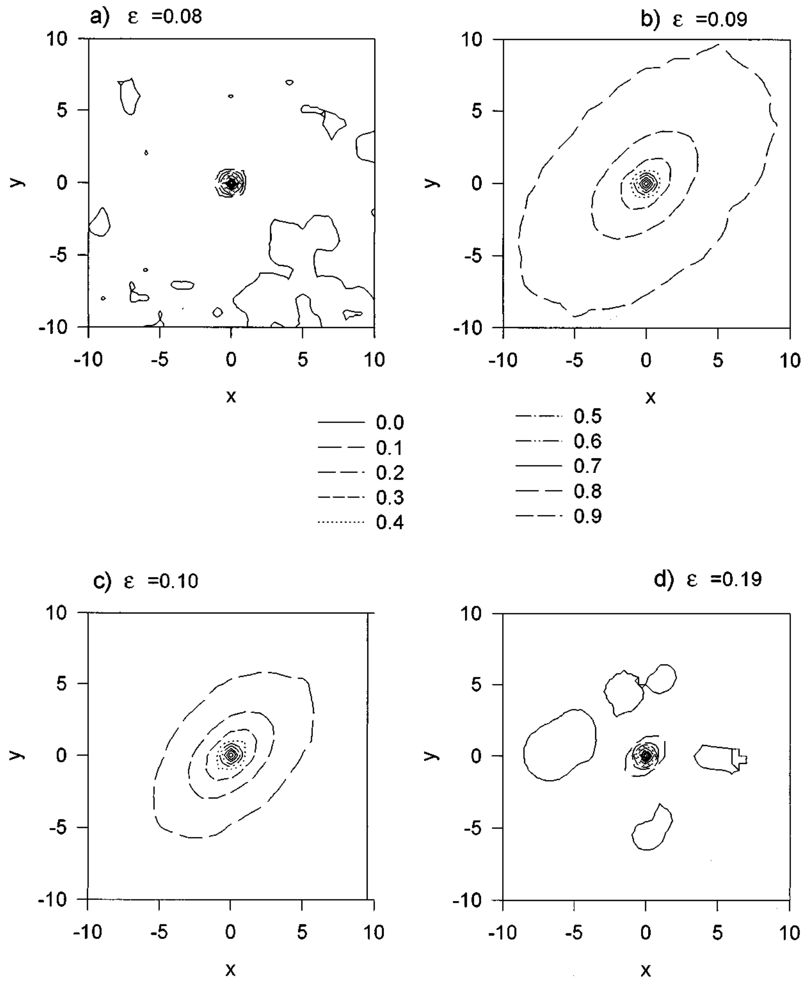


FIG. 1. Contour plots of the two-point correlation function corr $(\sigma_O,\sigma_{x,y})$ between some spin $\sigma_O$, the origin, and a spin $\sigma_{x,y}$ for different noise levels $\varepsilon$: (a) 0.08, (b) 0.09, (c) 0.10, and (d) 0.19. $x,y$ are horizontal and vertical coordinates of a spin $\sigma_{x,y}$, respectively, and they are measured in the lattice units. Notice the rapid change of properties at $\varepsilon$ crossing 0.090. $L=100$, the thermalization time is equal to 5000 time steps, and averages are made over 20 000 time steps.

The link between probabilistic cellular automata stationary evolution and equilibrium systems in the *space* $\times$ *time* lattice has been rigorously found. In this case the local Hamiltonian is expressed by the logarithm of the transition probabilities (see, e.g., [9,3]). It has been also found that all stationary measures of probabilistic cellular automata arising in the high-noise regime are Gibbsian [3]. In this regime the weakness of correlations between spins makes the probability of any configuration in a finite volume independent of distant spins [3].

However, there is still little known about the nature of stationary measures for systems apart from this regime [10]. Although the Gibbsian nature of the mentioned Domany cellular automata model has been proved rigorously [3], because of the presence of long-range correlations in Toom stationary configurations, there exists a suspicion that the locality of interactions can be violated and one cannot give any reasonable description for these interactions [10].

We observed correlations between two spins of the Toom model: the origin $O$ and its sub-sequent neighbors on a square lattice with respect to the noise level $\varepsilon$. The results are shown in Fig. 1. One can notice the critical change in the behavior of the two-point correlations with $\varepsilon$ crossing 0.09. Additionally, at this $\varepsilon$ value the average magnetization of the whole configuration rapidly goes down to zero. For coupled map lattices it is known that increasing the spatial correlation leads to the merging of new ground states [11]. Therefore, in the case of the Toom model, because of the two properties of long correlations and zero of the total magnetization, one can expect that the stationary state has changed into a mixture of homogeneous islands of two basic phases: the $(+)$ phase and $(-)$ phase (see [12] for details). Moreover, these properties give rise to the features of some equilibrium system even in the absence of a Hamiltonian energy.

## II. EXPERIMENTS AND RESULTS

### A. The two-point correlation function

First we want to determine $\varepsilon_{\mathrm{cr}}$, the critical value of the noise parameter at which this critical phenomenon takes place in the limit of infinite volume. The static critical behavior of any thermodynamic system in the infinite lattice limit can be extracted from the bulk properties at the critical point of finite systems [13,14]. The current resolution is such that it is reasonable to approximate the finiteness of the simulated lattice by [14]

$$\varepsilon_{\mathrm{cr}}(L)=\varepsilon_{\mathrm{cr}}+\lambda L^{-1/\nu} \tag{3}$$

where $\nu$ is one of the static critical exponents, and $\lambda$ is a constant dependent on the quantity from which the finite-lattice-size effect is extracted. Furthermore, it is known [14] that the following quantities also vary with the system size as $L^{1/\nu}$: the fourth-order magnetization cumulant $U$,

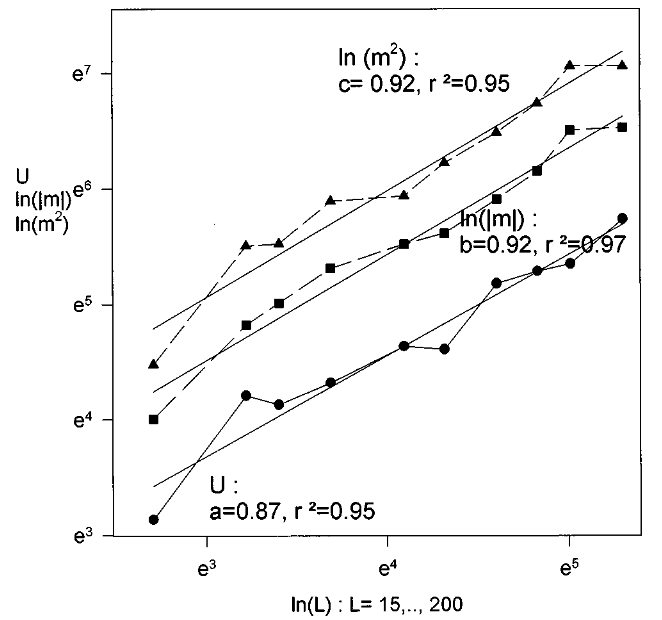


FIG. 2. A log-log plot of the lattice size dependence $L$ to the maximum values of derivatives for the cumultant $U$ of the absolute value of the spin magnetization $|m|$, the logarithm of this absolute value $\ln(|m|)$, and the logarithm of the square of spin magnetization $\ln(m^2)$ to estimate the finite lattice size influence. Some STD errors for this data are rather high, about 40%. The values of $a,b,c$ provide linear regression coefficients for the presented data with $r^2$ correlation coefficients for these fits.

$$U = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2},$$

where $m = L^{-d}\Sigma_i \sigma_i$ is the magnetization, and the logarithm of any power of the magnetization $m$, e.g., $\ln(|m|)$, $\ln(m^2)$. Thus the location of the maximum slope for $U$, $\ln(|m|)$, and $\ln(m^2)$ serves as an estimate of both $\nu$ and $\varepsilon_{\mathrm{cr}}(L)$.

Our results for $U$, $\ln(|m|)$, and $\ln(m^2)$ presented in Fig. 2 lead to $\nu \approx 0.90 \pm 0.02$. With $\nu$ determined we can estimate $\varepsilon_{\mathrm{cr}} \approx 0.091 \pm 0.002$ (at $\lambda \approx 0.07 \pm 0.01$).

It is interesting to ask about the character of the decay of the correlations with the increase of the spin distance. In Fig. 3 we present this decay along the horizontal line. To find out whether the decay is of power-law or exponential type we present results on a log-log plot [Fig. 3(a)] and a log plot [Fig. 3(b)]. For $\varepsilon \in (0.090, 0.100)$ we found that both numerical decay approximations, the exponential and power law, are faithful with respect to the accuracy of our simulation errors.

It is known that if the decay of correlations goes down as $|i-j|^{-\eta}$, where $|i-j|$ denotes distance between two spins and $\eta < 2$, this means that the system described is highly correlated with the Hamiltonian not in a quadratic form [9]. According to results obtained by us, such a complicated interaction can be considered as involved in stationary states of the Toom probabilistic model with $\varepsilon \in (0.090, 0.100)$, though the range of this interaction seems to not exceed 20 lattice units.

### B. Locality of interactions

The next experiment is designed to study the so-called long-range order, i.e., to verify how any finite volume con-

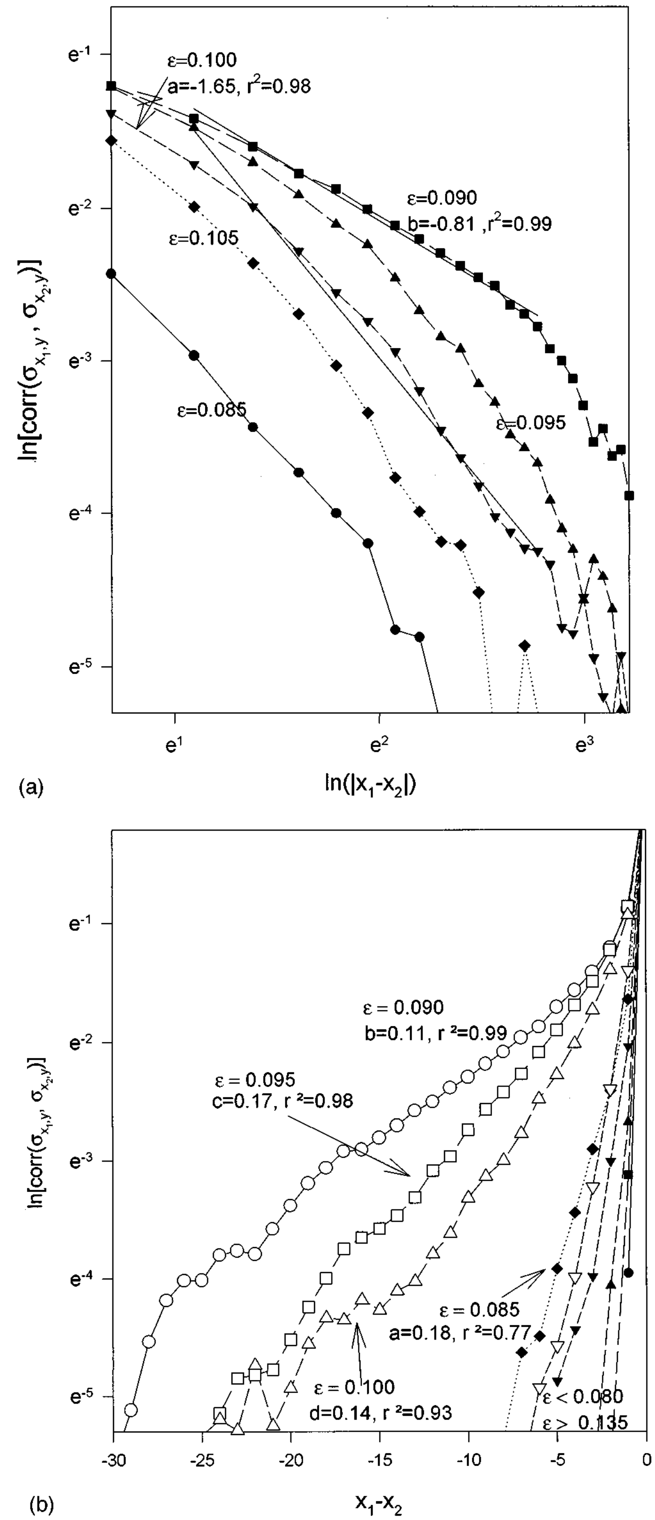


FIG. 3. (a) Power-law decay of correlations corr $(\sigma_{x_1,y}, \sigma_{x_2,y})$ between distant spins along the horizontal line (log-log plot). Values $a,b$ presented for the linear fits are found for $4 < |x_1 - x_2| < 15$. (b) Exponential decay of correlations corr $(\sigma_{x_1,y}, \sigma_{x_2,y})$ between distant spins along the horizontal line (log plot). The curves in (a) and (b) are labeled by the thermal noise $\varepsilon$. The values of $a,b,c,d$ provide linear regression coefficients for the corresponding data with $r^2$ the correlation coefficients for these fits.

figuration is conditioned by the distant surrounding boundary configuration [9,4,16]. For this purpose we observe properties of a nonhomogeneous system.

The nonhomogeneous system considered consists of two homogeneous but disjoint areas where one area, called *in*, is put inside the other area, called *out*. In such a system we test the influence of one homogeneity on the other one. The out configurations can be seen as boundaries for the in configurations. The experiment goes as follows. First, both configurations evolve individually to have both configurations thermalized at the corresponding noise levels $\varepsilon_{\rm in}$ and $\varepsilon_{\rm out}$ and with suitable initial phases. Then the in configuration is put to the middle of the out configuration. Then the next thermalization process begins. The inhomogeneity of dynamics is kept all the time. The results obtained after this doubled thermalization time, by means of the average of magnetization along a lattice, for different noise levels $\varepsilon_{\rm in}$ and $\varepsilon_{\rm out}$ and different initial out phases are shown in Fig. 4 [the initial phase of the in state is always $(+)$].

One can observe that the boundary state out determines the phase of the in state whenever both states are represented by a stationary state of one phase, i.e., $\varepsilon_{\rm in},\varepsilon_{\rm out} \ll \varepsilon_{\rm cr}$. The phase of the in state becomes the same as the out state, although there are two areas that are still different from each other by the magnetization level. This ''proves'' Gibbsianess nature of these stationary configurations [10].

The influence of the boundary when the states in and out are near or within the critical regime is not obvious. Follow, for example, the dotted lines in Figs. 4(a)–4(c), which represent states with $\varepsilon_{\rm out}=0.08$ (both initial phases) for distinct $\varepsilon_{\rm in}$. If the correlations between spins in the in states are significant, $\varepsilon_{\rm in}$ is about $\varepsilon_{\rm cr}$, then one can observe the effect caused by the periodic boundary conditions. In general, the periodic boundary conditions add a large scale of size $L$ into the system. Any configuration observed from this scale can be seen as the sea of the out configuration with infinite many regular islands of the in state. If the in state is weak, by means of correlating neighboring spin states, then each island acts in isolation. However, the strong correlating properties in the in state, together with good transmission properties of the out state, $\varepsilon_{\rm out} < \varepsilon_{\rm cr}$, yield that the in islands act together as the boundary for the out configuration. The islands can easily communicate with each other through the out state by adjusting the out state phase to their properties. Finally, we observe that the phase of the out configuration is chaotic, i.e., undetermined, and the level of its magnetization is random; see particular examples of such behavior in Fig. 4(d).

### C. Block magnetization experiments

The basic notion for the last group of experiments results from the so-called *wrong large-deviation properties* [4]. Namely, the probability that a configuration $\sigma$ taken from the probability distribution $\nu$ is inside a finite volume $\Lambda$, a typical configuration taken from some distribution $\mu$, decays exponentially in the volume of $\Lambda$ with rate $i(\mu|\nu)$, the relative entropy density of the measure $\mu$ with respect to the measure $\nu$. Therefore, by measuring this probability one can estimate the relative entropy density $i(\mu|\nu)$ as

$$i(\mu|\nu)=\lim_{\Lambda\to\infty} -\frac{1}{|\Lambda|}\ln(\mathrm{Prob}_{\nu}\{\sigma_{\Lambda}\text{is typical for}\mu\}). \tag{4}$$

Furthermore, if one found, e.g., via simulations, that

$$i_l(\mu|\nu) = \frac{1}{l^2}\ln(\mathrm{Prob}_{\mu}\{\text{all spins are up in a square } l\times l\} \to_{l\to\infty} 0, \tag{5}$$

then it would provide that $i(\delta_+|\mu)=0$, where $\delta_+$ is the measure concentrated on the all spin-up configuration. As $\delta_+$ is a non-Gibbsian measure, then $\mu$ is non-Gibbsian also [4]. We perform experiments at $\varepsilon \approx \varepsilon_{\rm cr}$ and observe the probability of finding totally magnetized squares. Unfortunately, the events with blocks $l\times l$ totally magnetized appear so seldom with $l$ growing that it is difficult for us to estimate the corresponding probabilities. Therefore, we allow ourselves to violate a little the total magnetization demand by setting it to $|m|>0.9$ (for better illustration other properties of the state we also present probabilities to meet a block with magnetization $|m|>0.8,0.6,0.4$, suitable). They results of these block distributions are presented in Fig. 5(a). They rather stable their values for $i_l$ with $l$ growing for blocks almost complete magnetized and slowly decrease in the case of less magnetized blocks. However, because of the small amount of data, this suggestion needs more verification.

If $\varepsilon < \varepsilon_{\rm cr}$, then we can look for the $(-)$ phase in the $(+)$ phase by estimating $i(\mu_{(-)}|\mu_{(+)})$. If $i(\mu_{(-)}|\mu_{(+)})>0$, then both measures must be non-Gibbsian because they are stationary measures of the same interactions [10]. The results for $i_l(\mu_{(-)}|\mu_{(+)})$ are presented in Fig. 5(b). According to them, the probability to find blocks with negative magnetization very slowly decreases with $l$ growing.

## III. CONCLUSION

We have examined properties of stationary states in Toom cellular automata that are in the regime of the second-type phase transition. First, by varying the temperature parameter $\varepsilon$ from $\varepsilon=0$ to $\varepsilon=1/2$, at $\varepsilon\in(0.09,0.10)$ we observe how stationary states of the model change their basic property: how these states transform from the states of either the $(+)$ phase or the $(-)$ phase into the equivalent mixture of two phases.

Toom interactions have a so-called eroder property [1]; it denotes that any finite island of one phase is smashed by the surrounding sea of the other phase. Therefore, stationary states that are representatives of one phase possess good transmission properties. After a short time the whole system takes the phase of the out state. The dependence of a stationary state island on the boundary conditions is continuous, i.e. local, for islands that are of one phase. One can say that after the thermalization process, which means adjusting the spins of an island, the stationary state remains the low-temperature ground state of the Ising-type model.

However, if an island is of the stationary state with critical properties, then it can oppose the outside world. One can say that at some $\varepsilon$ the eroder property is turned out by the

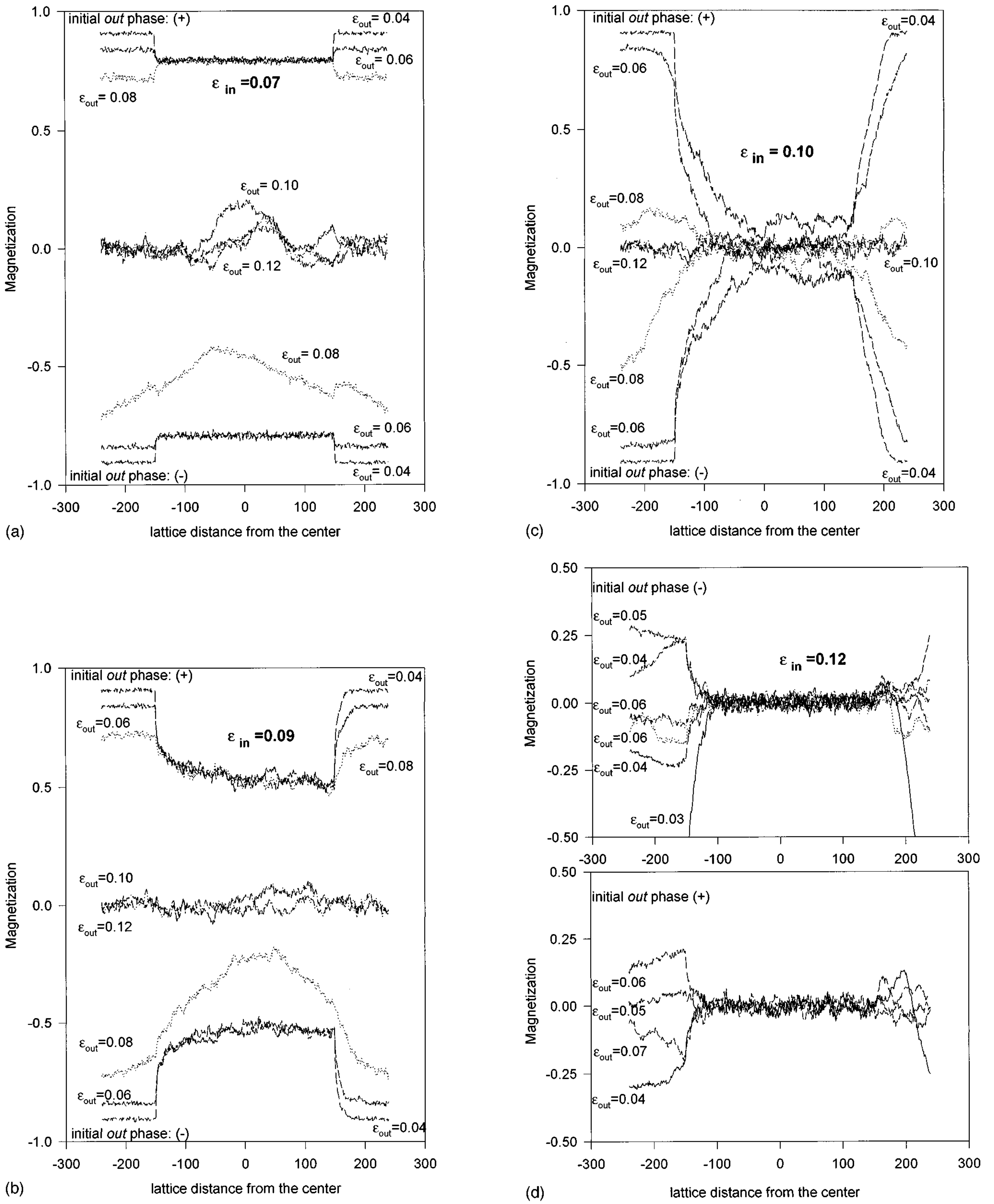


FIG. 4. Magnetization of a spin site along the lattice. The total size of the lattice $L=500$; the inner lattice size $L_{\mathrm{in}}=300$, $\varepsilon_{\mathrm{in}}=0.07$ (a), 0.09 (b), 0.10 (c), and 0.12 (d); and different levels of $\varepsilon_{\mathrm{out}}$ are denoted at the corresponding curves. The initial phase of *out* states is either $(-)$ or $(+)$, while the *in* state phase is always $(+)$. The averages are made over 30 000 time steps. 0 denotes the center of the lattice.

temperature effects and an island can survive. One might compare this observation with the notion of the balance between the production of local errors and the transmission strength that was considered by Boldrighini *et al.* [15] as the necessary condition for Ising-type transitions in coupled map lattices. Thus properties of stationary states for the Toom model considered with $\varepsilon > \varepsilon_{\mathrm{cr}}$ are generated by the extremely fragile balance present at the microscale. One can compare

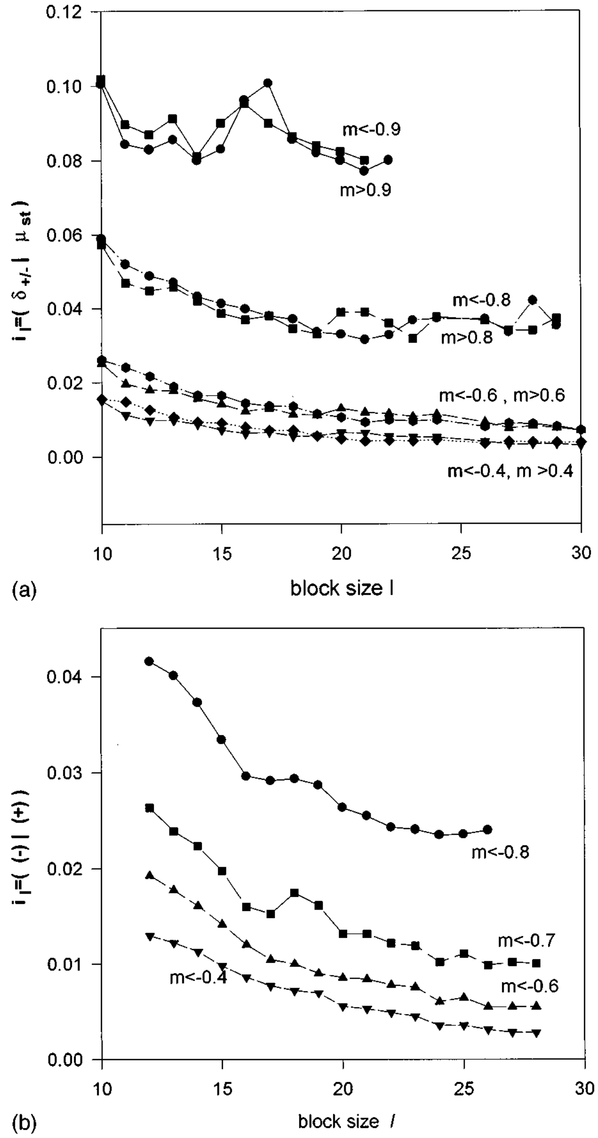


FIG. 5. Decay of $i_l(\mu|\nu)$ with growing size $l$ of a square block $l\times l$ to estimate the relative entropy density for (a) $i_l(\delta_{\pm}|\mu_{\rm st})$, with $\delta_{\pm}$ the measure with respect to the stationary measure $\mu_{\rm st}$ at the critical regime $\varepsilon=0.095$, and (b) $i_l(\mu_{(-)}|\mu_{(+)})$ the ''amount'' of the $(-)$ phase measure within the stationary measure of the $(+)$ phase state at $\varepsilon=0.090$. The curves are labeled by the different magnetizations of blocks considered.

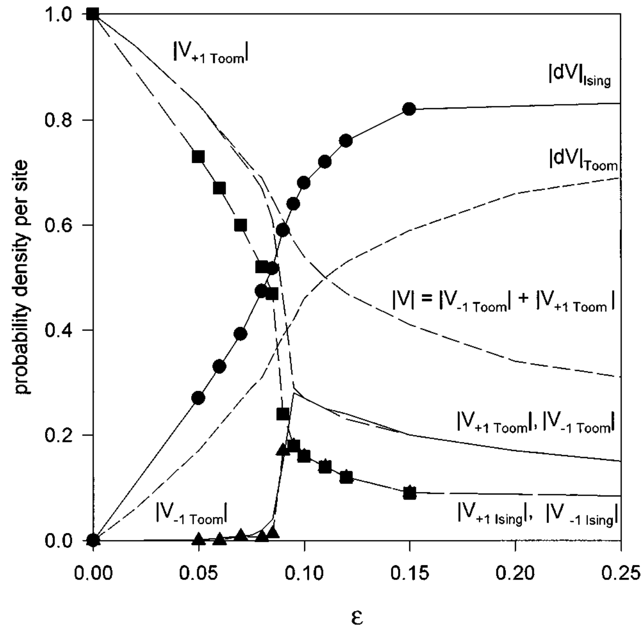


FIG. 6. Probability that a given site belongs to a cluster made of Toom neighborhoods (lines) or made of Ising neighborhoods (symbols and lines). The curves denoted $|V_{\rm Toom}|$ and $|V_{\rm Ising}|$ correspond to the energy density carried by stationary states at the given noise level $\varepsilon$. $L=200$ and the thermalization time is equal to 10 000 time steps.

this phenomenon with the condition of the detailed balance of equilibrium states.

The Toom model is worth considering as it recalls the Ising model. For the Ising model, the well-established Pirogov-Sinai theory [9,16] provides the tool for estimating energy of any finite volume configuration via the notion of a contour. The contour can be seen as the area separating homogeneous islands of pure phases. The local energy carried by the finite configuration is proportional to the total volume of contours that it contains. It is not difficult to estimate the energy carried by stationary states if one assumes only the definition of a site belonging to the cluster.

In the considered Toom model two such definitions can be taken into account. Namely, a site belongs to a cluster if its Toom neighborhood is homogeneous or a site belongs to a cluster if its standard Ising neighborhood, i.e., four nearest neighbors and the spin itself, is in the same spin state. Closing our investigations, we present in Fig. 6 the estimation for energy density in the case when clusters are made of either Toom or Ising neighborhood shapes. The probability that in a stationary state obtained at some $\varepsilon$, a given spin is in the up state and stays in it until the next time step, i.e., its site belongs to a $+1$ cluster, is represented by the volume of $+1$ clusters: $|V_{+1\rm Toom}|$ or $|V_{+1\rm Ising}|$. Corresponding to the two kinds of neighborhoods considered, the energy carried by a state is measured by the volume of spins not belonging to any cluster ($|dV_{\rm Toom}|$ or $|dV_{\rm Ising}|$ in Fig. 6).

The conclusion resulting is that stationary measures of the Toom model can mimic the measure of some equilibrium system, although strong subordering properties of the local rule makes this kind of diffusion process always present. Moreover, the slight difference between the parameter of the finite-lattice-size scaling $\nu=0.9$ found by us and $\nu=1$ proved rigorously for the two-dimensional Ising model, indicates a stronger influence of the finiteness of the lattice size in the Toom model than in the Ising model. However, the character of the correlation decay and results of large-fluctuation studies indicate that the stationary measure of the probabilistic Toom model is expressible in a Gibbsian form with respect to some, though possibly highly complicated, local Hamiltonian.

## ACKNOWLEDGMENTS

This research was partially supported by Polish Goverment Grant No. KBN 129/P03/95/09. Simulations were performed partially in the TASK-Academic Computer Center in Gdańsk, Poland.

[1] A. Toom, in *Topics of Contemporary Probability and its Applications*, edited by J. Kaurie Snell (CRC, Boca Raton, FL, 1995).
[2] E. Domany, Phys. Rev. Lett. **52**, 871 (1984).
[3] J. L. Lebowitz, Ch. Maes, and E. R. Speer, J. Stat. Phys. **59**, 117 (1990).
[4] A. C. D. van Enter, R. Fernández, and A. D. Sokal, J. Stat. Phys. **72**, 879 (1993).
[5] H. E. Stanley, D. Stauffer, J. Kertesz, and H. J. Herrmann, Phys. Rev. Lett. **59**, 2326 (1987); A. M. Mariz, H. J. Herrmann, and L. de Arcangelis, J. Stat. Phys. **59**, 1043 (1990); N. Jan and L. de Arcangelis, in *Annual Reviews of Computational Physics I*, edited by D.Stauffer (World Scientific, Singapore, 1994); P. Grassberger, Physica A **214**, 547 (1995).
[6] D. Makowiec, Physica A **234**, 435 (1996); Acta Phys. Pol. B **27**, 677 (1996).
[7] E. Domany and W. Kinzel, Phys. Rev. Lett. **53**, 311 (1984); O. Martin, Physica D **45**, 345 (1990); J. Miller and D. Huse, Phys. Rev. E **48**, 2528 (1993).
[8] F. J. Alexander, I. Edrei, P. L. Garrido, and J. L. Lebowitz, J. Stat. Phys. **68**, 497 (1992).
[9] Ya. G. Sinai, *Theory of Phase Transitions: Rigorous Results* (Academiai Kiadó, Budapest, 1982).
[10] Ch. Maes and K. Vande Velde, Physica A **206**, 587 (1994).
[11] L. A. Buminovich, A. Lambert, and R. Lime, J. Stat. Phys. **61**, 253 (1990).
[12] D. Makowiec, in *Stability of Ground States in Cellular Automata with Toom Local Rule*, Proceedings of the Eighth Joint EPS-APS International Conference on Physics Computing, edited by P. Borcherds, M. Bubak, and A. Maksymowicz (Academic Computer Center Cyfronet, Krakow, 1996), and unpublished.
[13] K. K. Mon and K. Binder, Phys. Rev. E **48**, 2498 (1993).
[14] D. P. Landau, in *Exploring Phase Transitions with Modern Monte Carlo Methods*, Proceedings of the Eighth Joint EPS-APS International Conference on Physics Computing (Ref. [12]).
[15] C. Boldrighini, L. A. Buminovich, G. Cosimi, S. Frigio, and A. Pellegrinotti, J. Stat. Phys. **80**, 1185 (1995).
[16] M. Zahradnik (unpublished).